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Geometrical aspects of Schlesinger's equation

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Abstract

The equation (Schlesinger's equation) for the isomonodromic deformations of an $SL(2, \mathbb{C})$ connection with four simple poles on the projective line is shown to describe a holomorphic projective structure on a surface. The space of geodesics of this structure is, by a primitive version of twistor theory, a two-dimensional complex Poisson manifold containing complete rational curves. The Poisson structure degenerates on a divisor and it is shown that the complement of the divisor is a symplectic manifold which can be identified with the quotient of the moduli space of representations of a free group on three generators in $SL(2, \mathbb{C})$ by the action of a braid group.

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Dedicated to André Lichnerowicz

1. Introduction

The duality between points and lines in the projective plane is a familiar notion. Perhaps less well-known is a nonlinear analogue of this, which forms one of the basic examples of twistor theory. From this point of view, the differential geometry of a holomorphic projective structure on a two-dimensional manifold U is encoded in the complex geometry of a surface U^* containing rational curves. The surface is the space of geodesics of the projective structure, and each rational curve in it consists of the geodesics passing through a particular point in U.

The purpose of this paper is to adopt this approach for a much-studied differential equation – the 4-point case of Schlesinger's equation. This equation appears in a multitude of places in mathematics and physics. It is equivalent to Painlevé's sixth equation and can be found in many aspects of integrable systems. The equivalence also shows that it is too much to ask for explicit solutions in general: these need the Painlevé transcendants.

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What we shall do here is to show that Schlesinger's equation describes a projective structure in two dimensions, and try and identify the complex surface U^* to which this corresponds in the twistor picture. The key to this identification is the isomonodromic interpretation of the equation: solutions are parametrized by representations of a free group on three generators, the fundamental group of a four-times punctured sphere. The mapping class group (here a braid group) acts on this space of representations by outer automorphisms of the free group, and we shall see that a quotient by this action is isomorphic to the complement of a divisor in the complex surface U^*.

Turning the question around, we learn that, after dividing by the mapping class group, the space of representations admits an extension to a complex manifold which contains compact rational curves. This is very close in flavour to the Mumford–Deligne compactification of the quotient of Teichmüller space by the mapping class group.

It is well-known that spaces of representations of surface groups are symplectic manifolds, and the symplectic point of view permeates this discussion. We show in particular that the symplectic structure on the space of representations extends to a naturally defined Poisson structure on the complex surface U^*.

2. Projective structures

Given an affine connection on a manifold M, we can define the concept of geodesics. These are distinguished curves with the property that through any point and in any direction there is a unique geodesic. When we say this, we think of geodesics as unparametrized curves, but the equations they satisfy, namely,

$$\frac{\mathrm{d}^2x_i}{\mathrm{d}t^2}+\sum_{j,k}\Gamma^i_{jk}\frac{\mathrm{d}x_j}{\mathrm{d}t}\frac{\mathrm{d}x_k}{\mathrm{d}t}=0, \tag{1}$$

provide a parameter t, well defined up to translation, on each curve. Two affine connections are *projectively equivalent* if the geodesics are the same curves, but with different parametrizations. This is equivalent to saying that the geodesic flows project to the same one-dimensional foliation on the projectivized tangent bundle $P(TM)$. In analytical terms it means that

$$\tilde{\Gamma}^i_{jk}-\Gamma^i_{jk}=a_j\delta^i_k+a_k\delta^i_j$$

for some 1-form $\sum a_i\,\mathrm{d}x_i$. A projective equivalence class of connections is called a *projective structure* on M.

When M is two-dimensional with local coordinates (x, y) the parameter t can be eliminated from (1) to give a nonlinear second-order ordinary differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\Gamma^1_{22}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+(2\Gamma^1_{12}-\Gamma^2_{22})\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+(\Gamma^1_{11}-2\Gamma^2_{12})\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\Gamma^2_{11}.$$

Conversely, as discussed in [2], any differential equation of the form

$$\frac{d^2y}{dx^2} = a_3\left(\frac{dy}{dx}\right)^3 + a_2\left(\frac{dy}{dx}\right)^2 + a_1\left(\frac{dy}{dx}\right) + a_0, \tag{2}$$

where the coefficients a_i are functions of (x, y), defines a projective structure.

We have implicitly assumed that we were working over the reals with C^∞ functions here, but the differential geometry makes just as good sense on a complex manifold M^n. There is, however, in this case an alternative treatment based on the study of complex submanifolds of complex manifolds – an elementary version of twistor theory – as described in [4,6]. In this version $P(TM)$ is a *complex* manifold with a foliation by the orbits of the geodesic flow of any representative affine connection. For a geodesically convex neighbourhood $U \subseteq M$, the quotient space U^* of the foliation is a well-defined complex manifold of dimension $2n - 2$ with projection map

$$\pi: P(TU) \to U^*$$

Each projectivized tangent space $P(T_xU)$ maps to a projective space $\mathbb{P}^{n-1}$ in U^*.

In dimension 2, U^* is a complex surface, and each projective space is a projective line with self-intersection number 1. The remarkable fact is that this information – a complex surface with a family of projective lines of self-intersection 1 – is sufficient to recover the projective structure on U. The normal bundle of each line is $O(1)$, and it is a consequence of a theorem of Kodaira that the lines are parametrized by a two-dimensional complex manifold. Given such a surface U^*, we simply define U to be this manifold – the locally complete family of lines in U^*. For each $\xi \in U^*$, the lines passing through ξ define a distinguished curve in U which is a geodesic of the projective structure.

This construction generalizes the duality between lines in a projective plane and points in the dual plane, but there is an essential difference. The space U has a local differential-geometric structure – a projective structure. The dual space U^*, the space of geodesics in U, is locally trivial – just a complex surface – but has globally defined projective lines lying in it.

Nevertheless, the basic features of duality remain. A point $x \in U$ defines a curve C_x in U^* – the projective line consisting of all the geodesics passing through x. A geodesic $C \subset U$ defines a point $\xi_c \in U^*$. If $x \in C$, then $\xi_c \in C_x$.

From the point of view of the differential equation (2), the graph of a solution of the equation is a geodesic in U. Graphs have the property that dy/dx is finite, and so during the elimination process of replacing t by x we have omitted geodesics for which $dx/dt = 0$ at some point. The space of geodesics U^* is thus covered by two open sets: one the solutions to (2), and the other solutions to the equation of the same form obtained by taking y as the independent and x the dependent variable.

We shall analyse this correspondence in the case of a particularly important nonlinear equation: Schlesinger's equation.

3. Schlesinger's equation

Consider the meromorphic connection defined on a rank m trivial vector bundle over $\mathbb{C} \times V$ (where $V \subset \mathbb{C}^n$ is an open subset) by the connection form

$$A = \sum_{k=1}^{n} A_k \frac{\mathrm{d}z - \mathrm{d}x_k}{z - x_k}, \tag{3}$$

where $A_k : V \to \mathfrak{gl}(m, \mathbb{C})$ is holomorphic. The flatness of the connection can be expressed as

$$\mathrm{d}A_i + \sum_{j \neq i} [A_i, A_j] \frac{\mathrm{d}x_i - \mathrm{d}x_j}{x_i - x_j} = 0 \tag{4}$$

and this *Schlesinger's equationm*:

We shall only consider the special case where $n = 3$ and $m = 2$: we take the A_k to be 2×2 matrices of trace zero, elements of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. By a projective transformation we can assume that $(x_1, x_2, x_3) = (0, 1, t)$ and then the equation can be written as

$$\begin{aligned} \frac{\mathrm{d}A_1}{\mathrm{d}t} &= \frac{[A_3, A_1]}{t}, \qquad \frac{\mathrm{d}A_2}{\mathrm{d}t} = \frac{[A_3, A_2]}{t - 1}, \\ \frac{\mathrm{d}A_3}{\mathrm{d}t} &= \frac{[A_1, A_3]}{t} + \frac{[A_2, A_3]}{t - 1}, \end{aligned} \tag{5}$$

where the last equation is equivalent to

$$A_1 + A_2 + A_3 = -A_4 = \text{const.}$$

Clearly the equation is unchanged by an overall conjugation of the matrices A_k by a constant matrix P. Notice also that for $1 \leq k \leq 4$

$$\frac{\mathrm{d}A_k}{\mathrm{d}t} = [A_k, C_k(A, t)]$$

for some matrix C_k so that as t evolves, each A_k lies on the same orbit $\mathcal{O}_k \subset \mathfrak{sl}(2, \mathbb{C})$. The integral curves of Schlesinger's equation thus lie on the subspace

$$\{(A_1, A_2, A_3, A_4) \in \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 \times \mathcal{O}_4 : A_1 + A_2 + A_3 + A_4 = 0\},$$

a five-dimensional space (if none of the A_k is zero). By invariance under overall conjugation we get an equation on the two-dimensional quotient M by the group $SL(2, \mathbb{C})$. (We assume here that we are dealing with an open set of stable points on which the action has a good quotient.)

To find local coordinates on this quotient consider the functions $x = \mathrm{tr}(A_1 A_3)$ and $y = \mathrm{tr}(A_2 A_3)$, which are well defined on M.

Lemma 1. *The functions x, y, are local coordinates in a neighbourhood of a point such that* $\mathrm{tr}\,(A_1[A_2, A_3]) \neq 0$.

Proof. A tangent vector to the product of orbits is of the form

$$(\dot{A}_1, \dot{A}_2, \dot{A}_3, \dot{A}_4) = ([A_1, B_1], [A_2, B_2], [A_3, B_3], [A_4, B_4])$$

and so $\mathrm{d}x$ and $\mathrm{d}y$ annihilate such a vector if and only if

$$\mathrm{tr}([A_1, B_1]A_3) + \mathrm{tr}(A_1[A_3, B_3]) = 0,$$
$$\mathrm{tr}([A_2, B_2]A_3) + \mathrm{tr}(A_2[A_3, B_3]) = 0$$

or equivalently

$$\mathrm{tr}((B_1 - B_3)[A_3, A_1]) = 0, \qquad \mathrm{tr}((B_2 - B_3)[A_3, A_2]) = 0.$$

If $\mathrm{tr}(A_1[A_2, A_3]) \neq 0$, then $[A_3, A_1] \neq 0$ and $[A_3, A_2] \neq 0$, so as we are working in $\mathfrak{sl}(2, \mathbb{C})$,

$$B_1 - B_3 = aA_1 + a'A_3, \qquad B_2 - B_3 = bA_2 + b'A_3. \tag{6}$$

Now since $A_1 + A_2 + A_3 + A_4 = 0$, we must have

$$\dot{A}_1 + \dot{A}_2 + \dot{A}_3 + \dot{A}_4 = [A_1, B_1] + [A_2, B_2] + [A_3, B_3] + [A_4, B_4] = 0$$

and hence substituting from (6),

$$[A_1 + A_2 + A_3, B_3] + a'[A_1, A_3] + b'[A_2, A_3] - [A_1 + A_2 + A_3, B_4] = 0.$$

Multiplying by $A_1 + A_2 + A_3$ and taking the trace,

$$a'\mathrm{tr}(A_2[A_1, A_3]) + b'\mathrm{tr}(A_1[A_2, A_3]) = 0,$$

and since $\mathrm{tr}(A_1[A_2, A_3]) \neq 0$, we have $a' = b'$. Thus from (6), $[A_1, B_1] = [A_1, B_3 + a'A_3]$ and $[A_2, B_2] = [A_2, B_3 + b'A_3] = [A_2, B_3 + a'A_3]$. Since clearly $[A_3, B_3] = [A_3, B_3 + a'A_3]$, we see that the tangent vector is tangential to the orbit of the overall $SL(2, \mathbb{C})$ action, and so zero on the quotient. It follows that $\mathrm{d}x$ and $\mathrm{d}y$ are independent and hence provide local coordinates. □

Let us consider the projection onto M of the Schlesinger equations. From (5), we have the differential equation for x and y:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\,\mathrm{tr}(A_1A_3)}{\mathrm{d}t} = \frac{\mathrm{tr}(A_1[A_2, A_3])}{t-1},$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}\,\mathrm{tr}(A_2A_3)}{\mathrm{d}t} = \frac{\mathrm{tr}(A_2[A_1, A_3])}{t}. \tag{7}$$

We set $f(x, y) = \mathrm{tr}(A_1[A_2, A_3])$. This is easily seen to be expressible as a function of x and y. In fact, in $\mathfrak{sl}(2, \mathbb{C})$

$$(\mathrm{tr}([A_1, A_2]A_3))^2 = -2\det\begin{pmatrix} \mathrm{tr}\,A_1^2 & \mathrm{tr}\,A_1A_2 & \mathrm{tr}\,A_3A_1 \\ \mathrm{tr}\,A_1A_2 & \mathrm{tr}\,A_2^2 & \mathrm{tr}\,A_2A_3 \\ \mathrm{tr}\,A_3A_1 & \mathrm{tr}\,A_2A_3 & \mathrm{tr}A_3^2 \end{pmatrix}$$

and as the A_k lie on fixed orbits, we have $\mathrm{tr}(A_k^2) = \alpha_k$, a constant. Together with

$$\begin{aligned}\alpha_4 &= \mathrm{tr}(A_4^2) = \mathrm{tr}(A_1 + A_2 + A_3)^2 \\ &= \alpha_1 + \alpha_2 + \alpha_3 + 2\mathrm{tr}(A_1A_2 + A_2A_3 + A_3A_1)\end{aligned}$$

this gives the explicit formula

$$f(x, y)^2 = -2\det\begin{pmatrix} \alpha_1 & \beta - x - y & x \\ \beta - x - y & \alpha_2 & y \\ x & y & \alpha_3 \end{pmatrix}$$

with $2\beta = \alpha_4 - \alpha_1 - \alpha_2 - \alpha_3$.

Differentiating (7) again we obtain

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{(f_x - 1)}{f}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{f_y}{f}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right), \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{(f_y + 1)}{f}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \frac{f_x}{f}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right).\end{aligned}$$

Since x, y are local coordinates we see that these are equations for the geodesics of an affine connection, and so we conclude:

Proposition 1. *Schlesinger's equation defines a projective structure on M.*

We can easily put the projective structure in the form of a second-order ordinary differential equation (ODE). From (7),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - t}{t} \tag{8}$$

and differentiating gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{t^2}\frac{\mathrm{d}t}{\mathrm{d}x},$$

so from (7) again we obtain the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{f}\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 1\right). \tag{9}$$

4. Poisson structure

Solving Eq. (9) explicitly is not an option: it is well known that this first case of Schlesinger's equation is equivalent to Painlevé's sixth equation, which can in general only be solved with new transcendental functions. Instead, we shall try to gain information about the equation by studying the space of solutions from the above point of view: the structure of U^* as a complex surface containing projective lines.

Consider first the tangent space to U^* at a solution $y(x)$ of (9). This consists of the vector space of solutions $u(x)$ of the linearized equation:

$$\frac{d^2u}{dx^2} + \frac{1}{f}\left(2\frac{dy}{dx}+1\right)\frac{du}{dx} - \frac{f_y}{f^2}\frac{dy}{dx}\left(\frac{dy}{dx}+1\right)u = 0.$$

Now any homogeneous second-order linear differential equation can be written in self-adjoint form

$$\frac{d}{dx}\left(F\frac{du}{dx}\right) - Gu = 0,$$

the consequence of which is that for any two solutions u_1, u_2 the skew-symmetric Wronskian expression

$$F\left(u_2\frac{du_1}{dx} - u_1\frac{du_2}{dx}\right)$$

is constant. This defines a 2-form ω on U^*, which may, however, be singular.

In fact the form *must* be singular at some points, as the geometry of the complex surface U^* shows. Recall that each projective line in U^* has self-intersection number 1. In other words, the first Chern class of the normal bundle N is 1. If K denotes the canonical bundle of the surface U^*, then restricted to each line C, we have

$$c_1(K) = c_1(K_C) - c_1(N) = -2 - 1 = -3$$

since the Euler characteristic of the projective line is 2. Thus any 2-form on U^* must have a pole on C of order 3. Let us consider what happens for Schlesinger's equation.

Here, from the linearization equation, we need the function $F(x)$ such that

$$\frac{F_x}{F} = \frac{1}{f}\left(2\frac{dy}{dx}+1\right).$$

But using the ODE (9), we find the solution

$$\frac{1}{F} = \frac{dy}{dx}\left(\frac{dy}{dx}+1\right)$$

and thus F, and hence ω, has singularities where $dy/dx = 0$ and -1. Each projective line in U^* is the projectivized tangent space of a point in U, and so this provides two points where the form has a pole. The third one can be seen by passing to the equivalent ODE with x expressed as a function of y, it is where $dx/dy = 0$.

The 2-form ω has poles in U^* on those solutions of (9) for which $dy/dx = 0, -1$ or ∞. By inspection these are the three sets of straight lines

$$y = a, \qquad y = -x + b, \qquad x = c.$$

As a, b, c vary they form a divisor $D \subset U^*$ with three components.

In two dimensions, a 2-form is a section of a line bundle – the canonical bundle K – and its inverse is a section of the anticanonical bundle K^{-1}. Thus in our case ω^{-1} is a holomorphic section of K^{-1} whose zero-set is the divisor D. Outside D, ω is non-vanishing and defines a symplectic form, which we shall call the *Wronskian form*.

A manifold with a section of $\Lambda^2 T$ whose Schouten bracket vanishes is a Poisson manifold. For the dual of a symplectic form the condition is automatically satisfied, so we can say:

Proposition 2. *The space of geodesics on a convex open set in M is a holomorphic Poisson surface.*

5. Isomonodromic deformations

In order to focus better on the space U^*, it is necessary now to consider the geometric origins of the Schlesinger equation.

Consider, then, the Riemann surface S consisting of $\mathbb{P}^1$ with $n+1$ disjoint discs D_i removed, one of them a neighbourhood of ∞. The fundamental group of S is a free group on n generators $\mu_1 \dots, \mu_n$, each one given by a choice of simple loop around the discs in the finite part. If we let μ_{n+1} denote the loop around infinity, the group is generated by $n+1$ elements with one relation.

$$\mu_1\mu_2\cdots\mu_{n+1} = 1.$$

The holonomy of a flat connection on S will then be defined by matrices $M_i \in GL(m, \mathbb{C})$ such that

$$M_1 M_2 \cdots M_{n+1} = 1. \tag{10}$$

If we fix the conjugacy classes C_i of each M_i, the space of matrices satisfying (10), and with no common invariant subspace, modulo conjugation by $GL(m, \mathbb{C})$ is a smooth manifold $\mathcal{M}$. If $n = 3$ and we consider connections with holonomy in $SL(2, \mathbb{C})$, the dimension of this manifold is $4 \times 2 - 2 \times 3 = 2$.

Now suppose in each disc D_i, we choose a point x_i. There is a straightforward way of writing down flat connections on S: consider the meromorphic 1-form

$$A = \sum_{k=1}^{n} A_k \frac{\mathrm{d}z}{z - x_k}$$

with $A_k \in \mathfrak{gl}(m, \mathbb{C})$. This defines a connection on $S = \mathbb{P}^1 \backslash D_1 \cup \cdots \cup D_{n+1}$, which is flat since A is holomorphic. If the eigenvalues of the residue A_k do not differ by a positive integer, a classical result says that the holonomy around the point x_k is conjugate to $\exp(-2\pi \mathrm{i} A_k)$. Thus if A_k lies in a fixed adjoint orbit $\mathcal{O}_k$, the holonomy M_k lies in a corresponding conjugacy class C_k. In this case the residues form a point

$$A_1, A_2, \dots, A_{n+1} \in \mathcal{O}_1 \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_{n+1}.$$

The sum of the residues of a meromorphic differential on a Riemann surface is always zero, so

$$\sum_{k=1}^{n+1} A_k = 0.$$

We form the quotient (of suitably stable points) by $GL(m, \mathbb{C})$. Our space M is just such a quotient, with $n = 3$ and $A_k \in sl(2, \mathbb{C})$, and again it is two-dimensional.

There is a relation between the two manifolds M and $\mathcal{M}$. For each $(x_1, \dots, x_n)$, the holonomy of the flat connection A defines a point in $\mathcal{M}$. In classical terminology, an *isomonodromic deformation* $A_k(x_1, \dots, x_n)$ is a family whose holonomy is fixed: it defines a single point in $\mathcal{M}$. Schlesinger's equation is the differential equation satisfied by the matrices A_k in an isomonodromic deformation.

Let us concentrate now on our basic example M with $n = 3$. In this case, for each $t \neq 0, 1, \infty$, and point $x \in M$ we have a connection form

$$A(t) = A_1 \frac{\mathrm{d}z}{z} + A_2 \frac{\mathrm{d}z}{z-1} + A_3 \frac{\mathrm{d}z}{z-t}.$$

We saw that a geodesic in M is an integral curve of the Schlesinger equation, which therefore corresponds to an isomonodromic deformation: it is determined by a fixed point $m \in \mathcal{M}$. Thus $\mathcal{M}$ parametrizes geodesics in M. Fix a point $x \in M$, then as t varies, we obtain a curve in $\mathcal{M}$, the holonomy of the connection $A(t)$. Each isomonodromic deformation of connections whose holonomy lies on the curve, passes through x: this is essentially a description of the duality discussed in Section 2, where now the isomonodromic deformations are geodesics of a projective structure.

It seems, then, that the representation space $\mathcal{M}$ is the complex surface U^* we are seeking. This is almost true, but there are two complications.

The first is that $\mathcal{M}$ is an affine variety, and can contain no compact projective lines. In some sense we see that already, each distinguished curve in $\mathcal{M}$ has parameter t restricted to lie in $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. Moreover, $\mathcal{M}$ is symplectic whereas U^* is Poisson. A more likely result is then that $\mathcal{M} = U^* \backslash D$.

It is the second complication which changes our point of view slightly: there is in fact no map from $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ to $\mathcal{M}$, only from the universal covering, which is the upper half-plane. The reason is that $\mathcal{M}$ was defined by using a particular set of generators of the fundamental group, giving holonomy matrices M_1, M_2, M_3, M_4. As t moves arounds 0,1 or ∞, the generators are changed by an element of the mapping class group – in this case the pure braid group $H(3)$. This is evident in a more abstract setting in Malgrange's derivation of the equations for isomonodromic deformations [7].

We obtain this way an injective map from $U^* \backslash D$ to $\mathcal{M}/H(3)$. The action on $\mathcal{M}$ is not everywhere proper and discontinuous, but then the restriction to an open set has already been made: we needed a geodesically convex open set $U \subset M$ to get a well-defined space of geodesics U^*. The result we obtain then is:

Proposition 3. *Let $U \subset M$ be a geodesically convex open set for the projective structure. Then there is an open set in the representation space $\mathcal{M}$ on which the braid group $H(3)$ acts freely and discontinuously and whose quotient is $U^* \backslash D$.*

6. Symplectic aspects

The relationship between $\mathcal{M}$ and $U^*\backslash D$ has another facet. Both are in a natural way symplectic manifolds, $\mathcal{M}$ because it is a moduli space of flat connections on a surface, and for that reason acquires a natural symplectic structure $\omega_{\mathcal{M}}$ [3]. Also, the space U^*, as we have seen, has a Poisson structure coming from the Wronskian on the space of solutions to the linearized ODE. We shall prove here the following.

Proposition 4. *Under the isomorphism above, the natural symplectic structure $\omega_{\mathcal{M}}$ on the moduli space $\mathcal{M}$ of flat connections coincides with the Wronskian on $U^*\backslash D$.*

Proof. This is a local question, and so the braid group complications do not appear. First consider the symplectic structure on $\mathcal{M}$. This is defined in the general case of a surface S with boundary by means of a moment map for a central extension of the group of gauge transformations acting on the affine space of connections $\mathcal{A}$ on S [1]

$$\mu: \mathcal{A} \to \hat{\mathfrak{g}}^*,$$
$$A \mapsto (F(A), A|_{\partial S}, 1),$$

where $F(A)$ is the curvature of the connection A. The boundary ∂S consists of a disjoint union of circles and a connection on a circle is determined up to gauge equivalence by its holonomy. It follows that if $\mathcal{O}$ denotes the space of connections on the boundary with fixed holonomy, then the moduli space $\mathcal{M} = \mu^{-1}(\{0\} \times \mathcal{O} \times \{1\})/\mathcal{G}$. The space $\mathcal{O}$ is a coadjoint orbit, and for this reason, we have a symplectic quotient and $\mathcal{M}$ inherits a symplectic form $\omega_{\mathcal{M}}$.

In the general situation of a symplectic manifold W with H-action, the symplectic structure on $\mu^{-1}(\mathcal{O})/H$ can be viewed as an ordinary symplectic quotient of the group H acting diagonally on $W \times \mathcal{O}$ with moment map J. The symplectic form on the symplectic quotient is then induced from the restricted 2-form on the submanifold

$$J^{-1}(0) = \{(x, \eta) \in W \times \mathcal{O}: \mu(x) = \eta\}. \qquad \square$$

In this gauge-theoretic case, the symplectic form is given as follows. If $\alpha, \beta \in \Omega^1(\partial S, \mathfrak{g})$ are two tangent vectors at a connection A, we take the symplectic form on $\mathcal{O}$

$$\omega_1(\alpha, \beta) = \frac{1}{2\pi \mathrm{i}} \int_{\partial S} \operatorname{tr}(\phi\beta),$$

where $\mathrm{d}_A\phi = \alpha$. On the space of connections $\mathcal{A}$ (which is the symplectic manifold W in our calculation) we have the symplectic form

$$\omega_2(\alpha, \beta) = \int_S \operatorname{tr}(\alpha \wedge \beta)$$

for $\alpha, \beta \in \Omega^1(S, \mathfrak{g})$. The symplectic form on $J^{-1}(0)$ is then

$$\omega = (\omega_1, \omega_2).$$

In our case, S has four discs removed. We take the orbit $\mathcal{O}$ to be the gauge orbit for which the holonomy is $\exp(-2\pi i A_k)$ around the kth boundary component, and consider

$$\nabla = d + A_1 \frac{dz}{z} + A_2 \frac{dz}{z-1} + A_3 \frac{dz}{z-t}.$$

For fixed t define a map $\tilde{\varphi}$ to $J^{-1}(0)$ by

$$\tilde{\varphi}(A_1, A_2, A_3) = (\nabla, \nabla|_{\partial S}, 1).$$

The map $\tilde{\varphi}$ descends to a map $\varphi : M \to \mathcal{M}$. We shall consider the pull-back of the canonical symplectic form on $\mathcal{M}$ to M.

Now M itself is a symplectic manifold, for the condition $A_1 + A_2 + A_3 + A_4 = 0$ is the vanishing of the moment map on the product of orbits $\mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 \times \mathcal{O}_4$. Quotienting by the action $SL(2, \mathbb{C})$ produces a symplectic manifold. We shall show in fact that the canonical symplectic form $\omega_{\mathcal{M}}$ on $\mathcal{M}$ pulls back to the canonical form ω_M on M. This is a useful intermediate step in identifying it with the Wronskian. The proof consists of showing that $\tilde{\varphi}^*(\omega_1, \omega_2)$ is the canonical form on $\mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 \times \mathcal{O}_4$.

Let B be a tangent vector at (A_1, A_2, A_3) to this product of orbits. Then

$$d\varphi(B) = [A_1, B_1]\frac{dz}{z} + [A_2, B_2]\frac{dz}{z-1} + [A_3, B_3]\frac{dz}{z-t}$$

for some B_k. For two such tangent vectors B, C with $\beta = d\varphi(B)$ and $\gamma = d\varphi(C)$ we have

$$\varphi^*\omega(B, C) = \int_S \operatorname{tr}(\beta \wedge \gamma) + \int_{\partial S} \operatorname{tr}(\phi\gamma).$$

Now the first term vanishes, since β and γ are (1,0)-forms. As to the second term, on each disc we can find a *holomorphic* $\phi(z)$ satisfying

$$\frac{d\phi}{dz} + [A_1, \phi]\frac{dz}{z} + [A_2, \phi]\frac{dz}{z-1} + [A_3, \phi]\frac{dz}{z-t} = \beta.$$

This is easily seen by expanding in power series around each singularity. Thus by Cauchy's residue theorem, the second term is

$$\int_{\partial S} \operatorname{tr}(\phi\gamma) = \sum_k \operatorname{tr}(-B_k[A_k, C_k])$$

and this is the canonical symplectic form on a product of orbits.

We have seen that, for each t, the function φ is a local symplectic diffeomorphism. We shall use this next to calculate the symplectic form on M in terms of the coordinates x, y. Denote the dependence of φ on t by φ_t. A solution curve of the Schlesinger equation is now given by

$$\{x(t) \in M : \varphi_t(x(t)) = \xi\}$$

for a fixed $\xi \in \mathcal{M}$; for, this is simply the definition of an isomonodromic deformation. Since φ_t is symplectic, differentiating with respect to t, we obtain a Hamiltonian vector field, and we deduce that the Schlesinger flow is Hamiltonian with respect to the canonical symplectic structure. From (5), the equations are:

$$\frac{\mathrm{d}A_1}{\mathrm{d}t} = \frac{[A_3, A_1]}{t}, \qquad \frac{\mathrm{d}A_2}{\mathrm{d}t} = \frac{[A_3, A_2]}{t-1},$$
$$\frac{\mathrm{d}A_3}{\mathrm{d}t} = \frac{[A_1, A_3]}{t} + \frac{[A_2, A_3]}{t-1}.$$

Using the canonical symplectic form, the Hamiltonian is easily seen to be

$$H = \frac{\operatorname{tr}(A_3A_1)}{t} + \frac{\operatorname{tr}(A_3A_2)}{t-1} = \frac{x}{t} + \frac{y}{t-1} \tag{11}$$

in the coordinates x, y on M. The Schlesinger equations in these coordinates are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{f}{t-1}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{f}{t}$$

and these are obtained by integrating the time-dependent vector field

$$X = \frac{f}{t-1}\frac{\partial}{\partial x} - \frac{f}{t}\frac{\partial}{\partial y}.$$

If the Hamiltonian is (11), the symplectic form must therefore be

$$\omega_M = \frac{1}{f}\,\mathrm{d}x \wedge \mathrm{d}y, \tag{12}$$

since

$$\iota(X)\omega_M = \frac{\mathrm{d}y}{t-1} + \frac{\mathrm{d}x}{t} = \mathrm{d}H.$$

Finally we relate this to the Wronskian symplectic form on $\mathcal{M}$. We again use φ_t for fixed t. For a solution of the Schlesinger equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-t}{t},$$

so fixing t is equivalent to fixing $\mathrm{d}y/\mathrm{d}x = (1-t)/t = c$. Now let ξ_1, ξ_2 be local coordinates on $\mathcal{M}$ and $y(x, \xi_1, \xi_2)$ the corresponding solution to (9). The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x, \xi_1, \xi_2) = c \tag{13}$$

defines x as a function of ξ_1, ξ_2, and so the local inverse ψ of φ_t is

$$\psi(\xi_1, \xi_2) = (x(\xi_1, \xi_2), y(x(\xi_1, \xi_2), \xi_1, \xi_2)).$$

Since we now know that $\psi^*\omega_M = \omega_{\mathcal{M}}$, we can relate this to the Wronskian. Differentiating y with respect to ξ_1 and ξ_2, we get solutions

$$u_1 = \frac{\partial y}{\partial \xi_1}, \qquad u_2 = \frac{\partial y}{\partial \xi_2}$$

of the linearized ODE, and differentiating the constraint (13) we have

$$\frac{d^2 y}{dx^2}\frac{\partial x}{\partial \xi_1} + \frac{du_1}{dx} = 0, \qquad \frac{d^2 y}{dx^2}\frac{\partial x}{\partial \xi_2} + \frac{du_2}{dx} = 0.$$

Thus

$$\begin{aligned}
\omega_{\mathcal{M}} &= \psi^* \omega_M \\
&= \psi^* \left(\frac{1}{f} dx \wedge dy \right) \\
&= \frac{1}{f} \left(\frac{\partial x}{\partial \xi_1}\frac{\partial y}{\partial \xi_2} - \frac{\partial x}{\partial \xi_2}\frac{\partial y}{\partial \xi_1} \right) d\xi_1 \wedge d\xi_2 \\
&= -\left(f \frac{d^2 y}{dx^2} \right)^{-1} \left(\frac{du_1}{dx} u_2 - \frac{du_2}{dx} u_1 \right) d\xi_1 \wedge d\xi_2 \\
&= \left(\frac{dy}{dx} \left(\frac{dy}{dx} + 1 \right) \right)^{-1} \left(\frac{du_1}{dx} u_2 - \frac{du_2}{dx} u_1 \right) d\xi_1 \wedge d\xi_2
\end{aligned}$$

from the ODE (9). This however is precisely the Wronskian form.

7. Conclusions

Schlesinger's equation has geometrical origins, in the problem of describing the holonomy of a flat connection. Formulated in the language of projective structures, its space of solutions can be extended, and in doing so we have seen how to extend the moduli space of flat connections, with its natural symplectic structure, to a Poisson manifold containing compact rational curves, at least after we have made the identification by using the mapping class group. The local differential geometry of the projective structure, and the fundamental fact that a geodesic emanates from each point in every direction, has given us this extension, but it must surely have a description in terms of flat connections too.

One possible scenario is that the extra points are equivalence classes of connections with only three simple poles. Note that as $t \to 1$, the poles of the connection

$$A_1 \frac{dz}{z} + A_2 \frac{dz}{z-1} + A_3 \frac{dz}{z-t}$$

with residues A_2 and A_3 approach each other. Since $dy/dx = (1-t)/t$, as $t \to 1$, $dy/dx \to 0$ and the limiting geodesics are of the form $y = \mathrm{tr}(A_2 A_3) = \alpha$, a constant. We may think therefore of replacing A_2 and A_3 in the limit by a common value $\tilde{A}_2$, and adding to $\mathcal{M}/H(3)$ the holonomy of a connection

$$\tilde{A}_1 \frac{dz}{z} + \tilde{A}_2 \frac{dz}{z-1}$$

with three singularities (including the point at infinity), with

$$\operatorname{tr}(\tilde{A}_1^2) = \alpha_1, \qquad \operatorname{tr}(\tilde{A}_2^2) = \alpha, \qquad \operatorname{tr}(\tilde{A}_3^2) = \alpha_4,$$

where $\tilde{A}_3 = -\tilde{A}_1 - \tilde{A}_2$. It would be interesting to know if this holds in any natural way.

We should finally point out that in [5] Hurtubise and Kamran approached the problem of studying solutions to the Painlevé equations by using the same twistor approach. They consider the second-order equation to be the Painlevé equation itself, however, and not the ODE (9) which is geometrically equivalent to it.

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